Schwarzschild 1916 seminal paper revisited :

A virtual singularity

G.D'Agostini[1] J.P.Petit[2]

Key Words : Schwarzschild, metric, central singularity, true singularity, coordinate singularity,virtual singularity, representation space, diabolo, torus

---

Abstract :

The seminal paper by K. Schwarzschild (1916) is revisited. The variable R introduced by the author in this paper was not the radial position but an auxiliary variable. However, during the following years, several authors (among them A. Einstein) have made a mistake using this variable as a true spatial coordinate leading to the prediction of a singularity that clearly do not really exist. The exact consequences of the model introduced by K. Schwarzschild hundred years ago are emphasized. Finally, we give the Schwarzschild's metric expressed in the set of the true (real-world) coordinates.

---

1 – **Introduction**

The present article is certainly going to be a surprise for many people. It is highly likely that the experts will not read it, no more than they read our article of 2015, simply due to its title " cancelllation of the singularity of the exchange Schwarzschild solution with natural mass inversion process. ". In what is going to follow we shall see that a very important piece of the General Relativity was based on a bad interpretation of the solution that Karl Schwarzschild brought to the equation of field introduced by Einstein, and published by him January 13th, 1915 in the Annals of the Academy of Science of Prussia [2].

2 – n**-surfaces defined by a Riemanan metric.**

Let us consider a surface which is described by n coordinates. We call metric a process allowing, by giving itself a point of the surface and a small travel onto this one, to calculate an element of length ds. Coordinates are real quantities. To be situated in the surface it is necessary that the ds is also real. In the case of a metrics Riemanienne the line element is:

(1) $$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu$$

Let's give a 2D example. Consider a torus:

---

[1] dagostinigilles@mailaps.org

[2] Jp.petit@mailaps.org

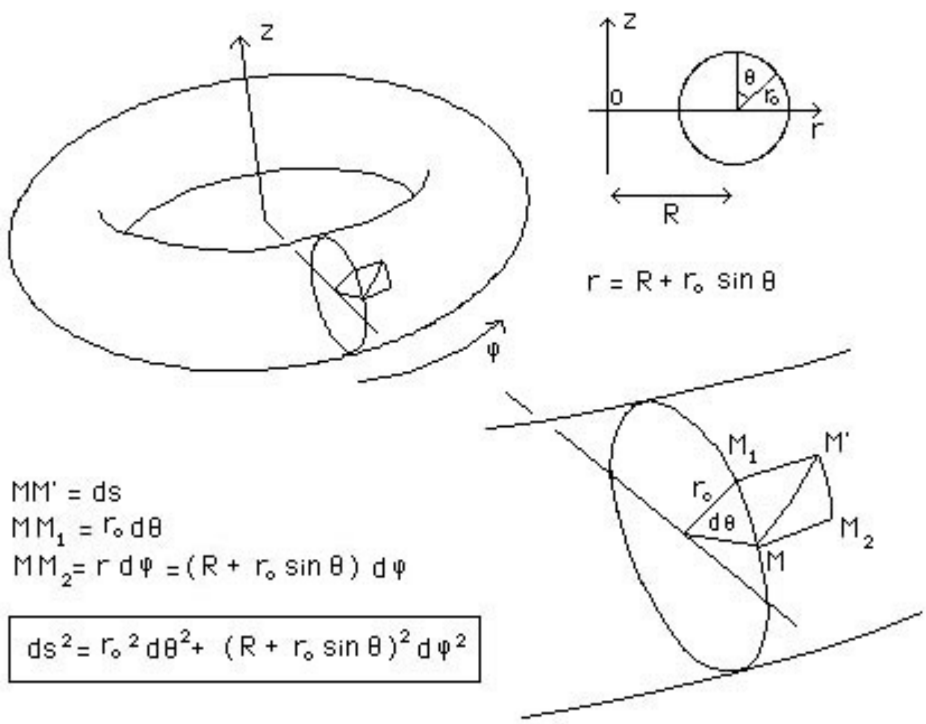

Fig.1 : Building the metric of the torus

We have chosen two angles θ and φ . We can present the metric as the following diagonal matrix :

(2)
$$g_{\mu\nu} = \begin{pmatrix} r_o^{\,2} & 0 \\ 0 & (R + r_o \sin\theta)^2 \end{pmatrix} = \begin{pmatrix} g_{\theta\theta} & 0 \\ 0 & g_{\varphi\varphi} \end{pmatrix}$$

$g_{\theta\theta}$ and $g_{\varphi\varphi}$ are the « potentials » of the metric. Considering the two signs of such elements, this set forms the « signature » of this Riemanian metric :

(3)
$$(+\ +)$$

With such coordinates these signs are always positive in such . The choice of coordinates, for example $\{\theta, \varphi\}$ corresponds to the choice of a space for representation (imaging space ?). Any couple of scalars $\{\theta, \varphi\}$ corresponds to a point of the torus.

A 2D-surface can be mapped through an infinite choice of coordinates. It is equivalent to the choice of two parametrized set of curves $C_1(p_1)$ and $C_2(p_2)$ forming a mesh

Coordinate singularities

The following example shows that the choice of the mesh system may induce artificial singularities. Consider the S2 sphere.

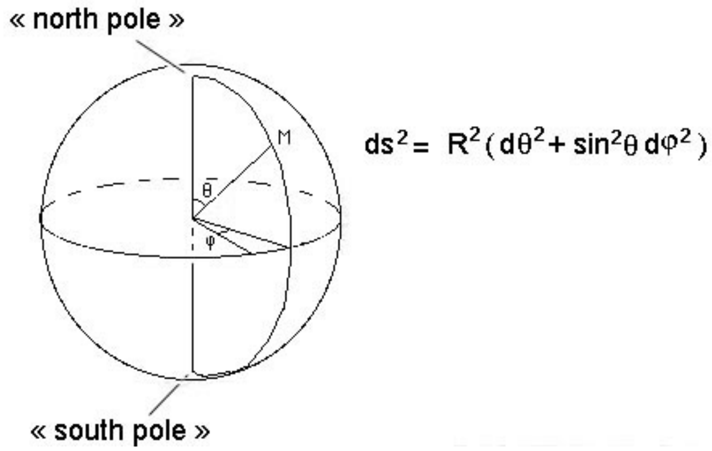

Fig.2 : Mapping the the S2-sphere and the corresponding expression of its metric

Here we have the classical system « longitude plus latitude ».

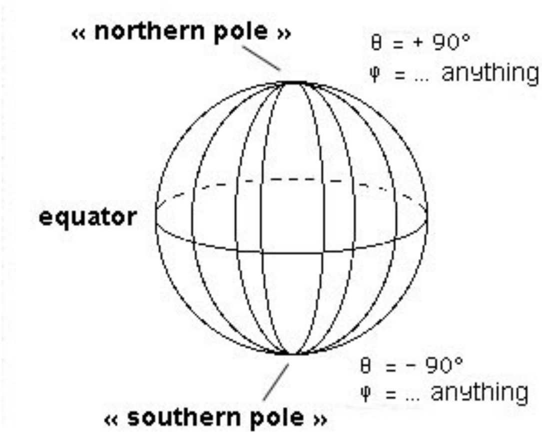

Fig.3 : Coordinate singularities on a S2-sphere

We could choose an infiniteway to map the S2-sphere. For example, see figure 4, with a single pole.

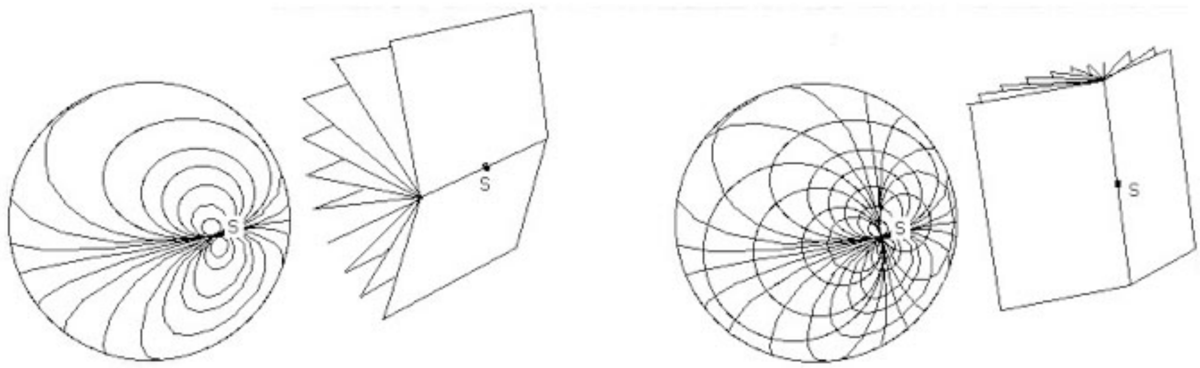

Fig.4 : Combining two families of curves we get a mapping with a single pole.

The precedent mapping of the torus had no coordinate singularity.

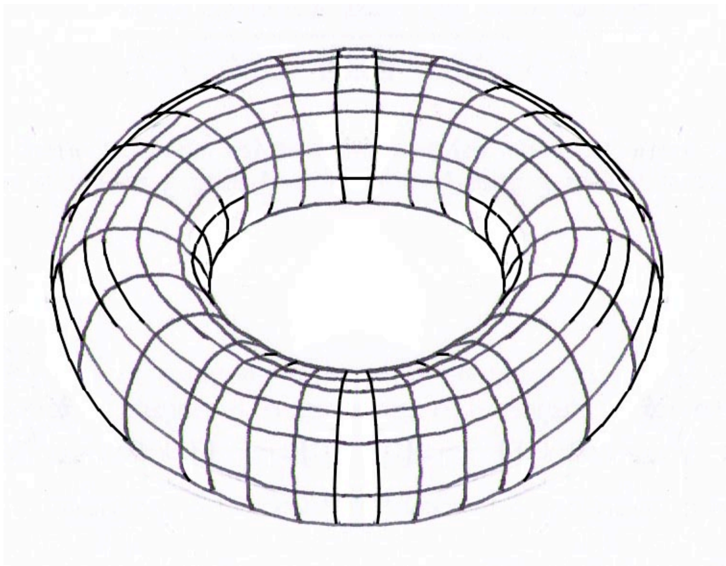

Fig.5 : Torus with (θ, φ) mapping

Focussing on this example of the torus we are going to create coordinates singularities. Introduce a coordinate r , which reprents the distance of a point to the axis of symmetry.

(4)

$$r = R + r_o \sin\theta \qquad\qquad \theta = \arcsin\left(\frac{r-R}{r_o}\right)$$

Then

$$d\theta = \frac{dr/r_o}{\sqrt{1-\left(\frac{r-R}{r_o}\right)^2}} = \frac{dr}{\sqrt{r_o^2 - R^2 - r^2 + 2rR}}$$

In this new coordinate system the metric becomes :

(5)

$$ds^2 = \frac{r_o^2\, dr^2}{r_o^2 - R^2 - r^2 + 2\,Rr} + r_o^2\, d\varphi^2$$

The denominator vanishes for :

$$r = \frac{R \pm \sqrt{R^2 - R^2 + r_o^2}}{1} = R \pm r_o$$

If we want the length element *ds* to be real, it implies :

(6)

$$R - r_o < r < R + r_o$$

Can the circles

$$r = R + r_o \qquad\qquad R = r - r_o$$

be considered as singular places ? No, there are coordinate singularities, *due to a wrong choice of coordinates.*

Let's give another example. Consider a 2-surface defined by its metric :

(7)

$$ds^2 = \frac{dr^2}{1 - \frac{R_s}{r}} + r^2\, d\varphi^2$$

When $r \to \pm\infty$ the metric tends to $ds^2 = dr^2 + r^2\, d\theta^2$ (euclidean space)

It shows a singularity at $r = R_s$. . Let's show this is a coordinate singularity. At first the signature, if $r > R_s$ i $(++)$.

In such domain the length element is real.

But what about the domain ($0 < r < R_s$) where the signature becomes $(-+)$ and where radial paths ($d\varphi = 0$) give imaginary length element ?

- When isometric embedding is possible

It is easy, as for the torus, to show that we are simple *out of the surface* ; In effect, in that peculiar case it is possible to deal with an *isometric embedding* (which preserves lengths) .

Our object, rotationnaly symmetric, can be embedded in R^3. Curves corresponding to the cut of the object by constant planes φ = cst, containing the axis of symmetry are meridian lignes. They correspond to

(8) $$ds^2 = dr^2 + dz^2 = \frac{dr^2}{1 - \frac{R_s}{r}} \Rightarrow dz = \frac{\pm\, dr}{\sqrt{\frac{r}{R_s} - 1}} = \pm\, 2\, R_s\, d\sqrt{\frac{r}{R_s} - 1}$$

(9) $$r = R_s \pm \frac{z^2}{4R_s}$$

It's a parabola. We can draw the surface, isometricaly imbedded in R^3. Call it a diabolo.

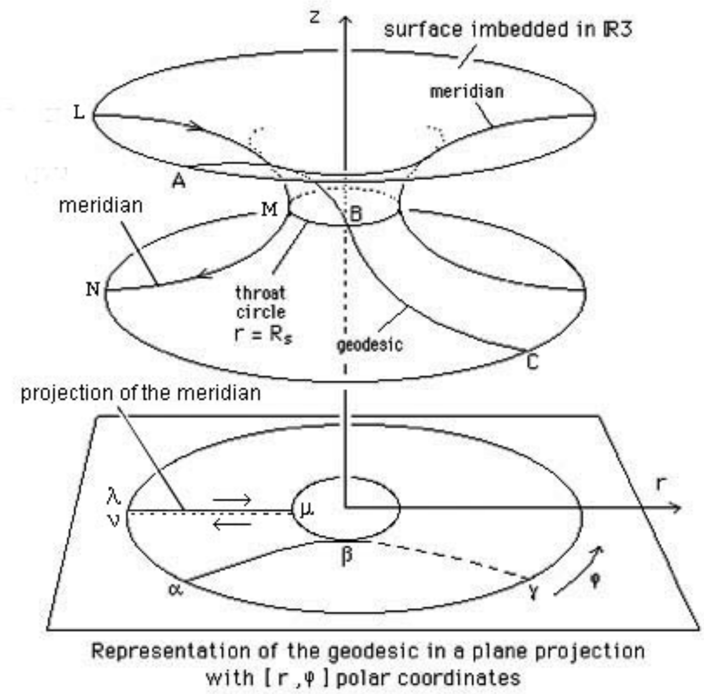

FiG. 6 : The diabolo

This geometric structure can be *imagined* (in the literal sens) as a link between, a to 2D euclidean spaces, two places : some sort of 2D space bridge. We may calculate the associated geodesic system :

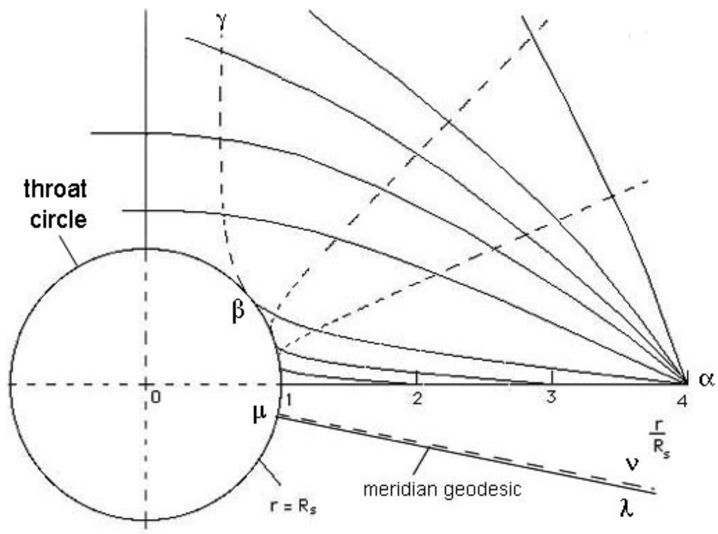

Fig. 7 : Geodesic system in a $\{ r , \varphi \}$ representation space.

But, in differential geometry, our goal is to study n-geometrical object without any reference to (n+1)-space where it could be imbedded in. In particular, the geodesics exist whatever the representation space is, how we choose to *imagine* it (to make an image of).

- What associated topology ?

Look back to the formula (7). It's a combination of letters. It's our choice to decide what is a variable and what is a constant, if they are positive or negative. We precise what are our choices. For example we decide that variables and length shall have real values. We can impose positive values for le scalar r , thinking about some « radial distance ».

We know that a n-surface, defined by its metric, has its own existence, and that the choice of coordinates just corresponds on how we want to figure the object. It's a *representation space* (imaging space).

When imbedding the object in an enclidean 3-space we see that positive r corresponds to two folds, instead a single one.

If we forget such 3-space, the topology of this 2D objects corresponds to the two folds cover of a plane plus a hole. The two folds are linked along their inner border, which becomes a throat circle.

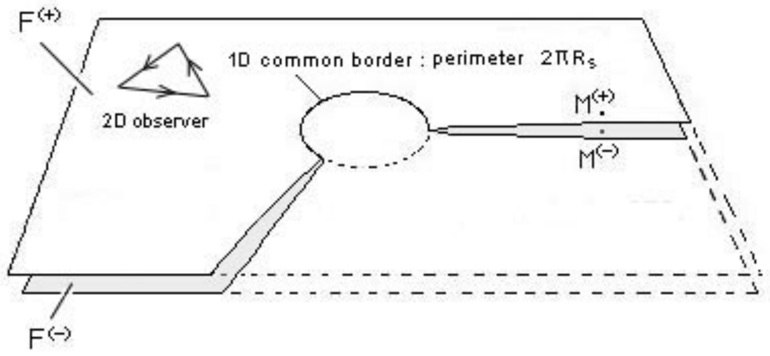

Fig. 8 : Two-fold cover of a manifold with a circle as common boundary

Such space is not null-homotopic, which means that we cannot deal with any closed loop drawn on the surface and make its length reduced to zero.

By the way, is it the only possible representation space ? Non. It depends on the choice of the coordinate system.

- The choice of the topoloy of the representation space.

Introduce the following change of coordinate :

(10) $$r = R_s\,(1 + Log\,\mathrm{ch}\,\rho\,)$$

The metric becomes :

(11)

$$ds^2 = R_s^2 \left[\frac{(\,1 + Log\,ch\,\rho\,)}{Log\,ch\,\rho}\, th^2\rho\ d\rho^2 + (1 + Log\,ch\,\rho\,)^2\ d\varphi^2 \right] = g_{\rho\rho}\, d\rho^2 + g_{\varphi\varphi} d\varphi^2$$

The function $ch\rho = \dfrac{e^{\rho} + e^{-\rho}}{2} \geq 1$ if well defined for any value of ρ .

$Log\ ch\rho \geq 0$ tends to zero when $\rho \to 0$. So, what about $g_{\rho\rho}$?

We can perform two expansion into Taylor's series.

$$e^{\rho} = 1 + \rho + \frac{\rho^2}{2} + \ldots.. \qquad e^{-\rho} = 1 - \rho + \frac{\rho^2}{2} + \ldots..$$

$$th\,\rho = \frac{e^{\rho} - e^{-\rho}}{e^{\rho} + e^{-\rho}} \simeq 0 \quad \text{when } \rho \to 0.$$

$$Log\ ch\ \rho \simeq \frac{\rho^2}{2} \quad \text{when } \rho \to 0.$$

So that $g_{\rho\rho} \to 2$ when $\rho \to 0$.

Phrased with such coordinates the metric is fully regular.

Let's imbed this geometric object in R³. Keep in mind that ρ is a number, not a length. Write the differential equation for meridians :

(12) $$ds^2 = R_s^2\, d\rho^2 + dz^2 = R_s^2\, \frac{(1 + Log\, ch\, \rho)}{Log\, ch\, \rho}\, th^2\rho\; d\rho^2$$

We can put Rs = 1

$$dz = \pm \sqrt{\frac{(1 + Log\, ch\, \rho)}{Log\, ch\, \rho}\, th^2\rho - 1}\;\; d\rho$$

$$\rho \to 0 \;\Rightarrow\; \frac{dz}{d\rho} \to 1 \qquad\qquad \rho \to \pm\infty \;\Rightarrow\; \frac{dz}{d\rho} \to 0$$

The meridian is :

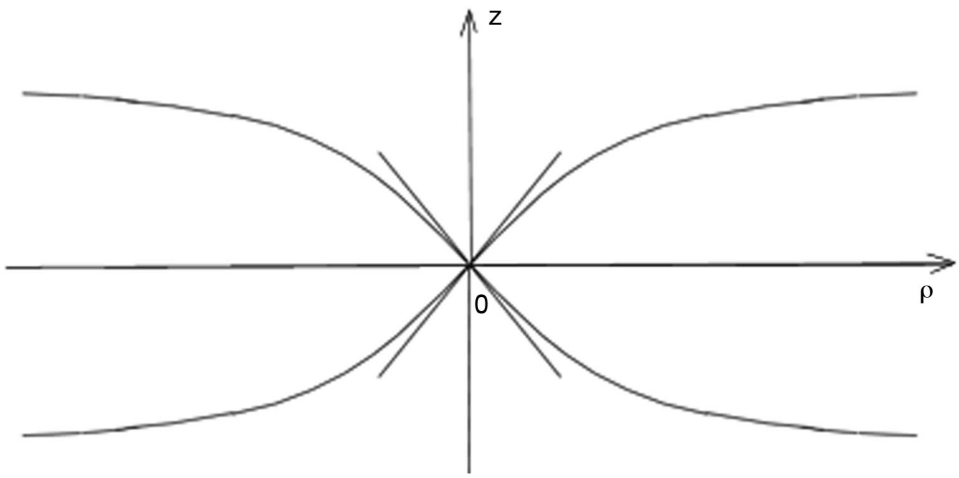

Fig.9 : Meridian lines in a ρ , φ representation

Then the geometrical object becomes some sort of cone, linking two planes. The surface becomes null homotopic : we can draw a closed loop on it and transform it to a point by a regular homotopy.

So, is the object defined by its metric (7) a cone or a space bridge ?

It depends what representation space you choose.

- Exploring 3D-hypersurfaces

With 3D-objects imbedding becomes impossible in our familiar 3D-euclidean representation space.

Consider the metric

(13) $$ds^2 = \frac{dr^2}{1 - \frac{R_s}{r}} + r^2 \,(\, d\theta^2 + \sin^2\theta \, d\varphi^2 \,)$$

Perhaps, now, you begin to realize towards what object I intend to take you. This line element is nothing but the space part of the Schwarzschild line-element.

We can apply the same coordinate change

(14) $$r = R_s \, (1 + Log\,ch\,\rho \,)$$

and get :

(15 $$ds^2 = R_s^2 \left[\frac{(\, 1 + Log\,ch\,\rho \,)}{Log\,ch\,\rho} \, th^2\rho \; d\rho^2 + (1 + Log\,ch\,\rho \,)^2 (\, d\theta^2 + \sin^2\theta \, d\varphi^2 \,) \right]$$

This metric is regular. When $\rho \rightarrow \pm\infty$ it tends to 3D euclidean space. You can imagine it as two structures.

In a $\{ \, r \, , \theta \, , \varphi \, \}$ *representation space* it becomes a 3D-space bridge, linking two 3D-euclidean space. This structure owns a throat sphere S2 who (minimum) area is $4 \, \pi \, R_s^2$

It is not null homotopic. If you consider a sphere that « contains » the thoat sphere you cannot contract ot to a point.

In a $\{ \, \rho \, , \theta \, , \varphi \, \}$ *representation* space it becomes a null homotopic 3D-cone .

- Now, let's deal with so-called black holes.

Fasten your sit belt.

In the following we are going to restricy ourself to spherically symmetric object, described by Schwarzschild metric. Axially symmetric objects, presented as so-called rotating black holes, corresponds to Kerr metric. This is somewhat different, but the conclusion is the same.

Where does Schwarzschild metric come from ?

In 1915 Einstein publish two papers ([1], [2]). One refers to his famous field equation and the other to an explanation of the advance of Mercury's perihelion. A careful examination of his paper shows that it is restricted to a weak field and brings almost newtonian paths. If *M* is the mass associated to the process, *c* the velocity of the light and *G* the constant of gravity, we can form the characteristic Schwarzschild's length :

(16) $$R_s = \frac{2\,G\,M}{c^2}$$

The delicate problem is to define what is a radial length, how we will give such mathematical results a physical interpretation.

Let's go back to the seminal Schwarzschild's paper [3]. You may have a look at it through an english translation at reference [3].

At firts Schwarzschild writes :

(17) $$ds^2 = F\,dt^2 - G\,(dx^2 + dy^2 + dz^2) - H\;(x\,dx + y\,dy + z\,dz)^2$$

To him, $\{\,x\,,y\,,z\,\}$ are what he calls rectangular coordinates. And he clearly defines a radial distance as :

(18) $$r = \sqrt{x^2 + y^2 + z^2} \ge 0$$

At the end of page 4 he introduces an *auxilliary quantity*, as defined by :

(19) $$R = (r^3 + \alpha^3)^{1/3} = (r^3 + R_s^3)^{1/3} \ge R_s$$

and he express his solution, using this auxilliary quantity, as :

(20)

$$ds^2 = (1 - \frac{\alpha}{R})dt^2 - \frac{dR^2}{1 - \dfrac{\alpha}{R}} - R^2\,(d\theta^2 + \sin^2\theta\,d\varphi^2) \qquad , \qquad R = (r^3 + \alpha^3)^{1/3}$$

or :

(20')

$$ds^2 = (1 - \frac{R_s}{R})dt^2 - \frac{dR^2}{1 - \dfrac{R_s}{R}} - R^2\,(d\theta^2 + \sin^2\theta\,d\varphi^2) \qquad , \qquad R = (r^3 + R_s^3)^{1/3}$$

This may seem familiar to the specialist in cosmology, but we have to note *that R is definitively not a radial distance*. From (19) *r* and *R* grow together. Moreover, at large distance :

(21) $$R = r\,(1 + \frac{R^3}{R_s^3})^{1/3} \quad R \to \infty \quad \Rightarrow \quad R \simeq r$$

But, at short distance :

(22) $$r \to 0 \qquad \Rightarrow R \to R_s$$

According to the way Scharzschild defines his variables it is not allowed to consider

$R < R_s$

As the matterfact, when we consider such values, the signature changes. Considering radial paths the length becomes imaginary.

Conclusion :

Such values correspond to points that are out of the hypersurface.

About the representation space , we have two choices. If we limit to $R > R_s$, the $\{t, R, \theta, \varphi\}$ representation corresponds to a space bridge linking two Minkowski spaces at infinite, through a throat sphere.

$R < R_s$ has no physical meaning

We can shift to a $\{t, r, \theta, \varphi\}$ representation space where r is the radial distance, as defined by Schwarzschild. Then, using the inverse transformation

(23) $$r = \sqrt{x^2 + y^2 + z^2} = (R^3 - R_s^3)^{1/3}$$

The, we get :

(24) $$ds^2 = \frac{\left(R_s^3 + r^3\right)^{1/3} - R_s}{\left(R_s^3 + r^3\right)^{1/3}} dt^2 - \frac{r^4}{\left(R_s^3 + r^3\right)\left(\left(R_s^3 + r^3\right)^{1/3} - R_s\right)} dr^2 - \left(R_s^3 + r^3\right)^{2/3} \left(d\theta^2 + \sin^2\theta \, d\phi^2\right)$$

This is not « a new metric », nor « a new solution of Einstein's equation ». This is nothing but the Schwarzschild's metric, expressed in the set of coordinates $\{t, r, \theta, \varphi\}$ he choose at the begining if his paper.

When $r \to +\infty$ this metric tends to Lorentz form.

(25) $$ds^2 = dt^2 - dr^2 - r^2 (d\theta^2 + \sin^2\theta \, d\varphi^2)$$

When $r \to 0 \quad \Rightarrow \quad g_{tt} \to 0$ *and* $g_{rr} \to 0$

To check if $r = 0$ is a singular point, let's calculate the Kretchmann scalar (invariant through coordinate change). In $\{t, R, \theta, \varphi\}$ this scalar is :

(26) $$K = \frac{48 G^2 M^2}{c^4 R^6} = \frac{12 R_s^2}{R^6} = \frac{12 R_s^2}{(r^3 + R_s^3)^2}$$

When $r \to 0 \qquad K \to \frac{12}{R_s^4}$

So that the point $r = 0$ (or $R = R_s$) *is not a singularity*

All that is presented above shows that we cannot think about a geometric object without using a *representation space*. The choice is up to us. Mais when we do that, it implies consequences about bordering and topology.

If we choose $\left\{ t\ , r\ , \theta\ , \varphi \right\}$ as a representation space (Schwarzschild's choice) it works for any positive values of r , from zero (included) to infinite, and we don't find any singularity.

We can draw the geodesic lines. Some end at the point $r = 0$, others turn around or have Newtonian like paths. Considering the geodesics that end at $r = 0$ it goes with *uncompleteness* of the geodesic system.

We can think about completing this geodesic system, extending the geodesic lines over the point $r = 0$.

How ?

We will see if Schwarzschild's calculation could be extended with the larger hypothesis :

(27) $$r = \pm \sqrt{x^2 + y^2 + z^2}$$

If it works, it would mean that the hypersurface could be extended to a $r < 0$ fold.

- What is physics ?

Physics goes with a representation space. In fact, this choice of the representation space gives our physical image of the world.

For example, we can decide that our world is real ? And of course it is. An imaginary world would correspond to some sort of metaphysics, which is out of the present scope.

In our mind we cannot use a mental image other than the brave old euclidean and null homotopic 3D-space . To handle any geometrical object we immediatly try to imbed it in that peculiar 3D space, where we believe we live in.

It is very difficult to imagine something like a space bridge (non null homotopic) or a null-homotopic 4D-cone (!...). Astrophycist have a great difficulty to deal with.

Mathematicians do not use mental imaging technique. They just play with letters, types, symbols, with syntaxic rules. It can be incredibely efficient, sometimes. Let us give an example. Consider a regular 2-surface (sphere, torus, Klein's bottle, Boy's surface). Using an homotopy we can transform these surface softly, in an euclidean space. They can be imbedded or immersed. In imbedding no self intesection is allowed. Immersions allow it, if the tangent plane varies continuously.

In 1957 Steven Smale showed, using just sympbols, that the 2S sphere owned a single homotopy group. His research director, Raoul Bott, was immediatly skeptical and said :

- *That cannot be true ! If it was true it should be possible to achieve the eversion of a sphere, to transform the standard sphere into its antipodale imbedding.*

But Smale, confident in his mathematical formalism insisted. Bott said

- *Show me how you manage it !*

Smale had absolutely no idea about that process and, to tell the truth, he did not care about it. Ten Years later my american friend Anthony Phillips found the first version and published it in Scientific American journal. Ugly, by the way.

The blind (…) french mathematician Bernard Morin found the second one. I drew it.

This an example where the mathematical formalism was incredibly in advance with respect to the common sense.

But, sometimes mathematicians and theoretician can build chimera. The Kruskal extension is one of those. It is supposed to be a technique that makes possible to explore the $0 < R < R_s$ region !

- Who did the mistake ?

After the Schwarzschild's publication [3] someone made a mistake. Who ? A careful exploration of the published papers will give the answer.

The mistake was to identify the intermediary variable $R = \left[(x^2 + y^2 + z^2)^{3/2} + R_s^{\,3} \right]^{1/3} \geq R_s$ to the radial distance $r = \sqrt{x^2 + y^2 + z^2}$.

The Kruskal analytic extension, « which makes possible to look inside a black hole » is a very beautiful mathematic building, but has nothing to do with physics, because it refers to a place which is out of the 4D-hypersurface. As a consequence, many objects become imaginary in this « inside », that only exists in the imagination of these theoreticians. This puzzling fact should pull their attention.

Some experts argue :

- *You neglect a portion of the hypersurface where the Ktreichmann scalar becomes infinite !*

Our answer :

- *You deal with something that is simply out of the hypersurface. Your analysis and theorems are not relevant.*

Physics corresponds to a choice of coordinates, ofsome representation space. The black hole model is founded on Schwarzschild and Kerr metrics. The behaviour of the object, in time, depends on the choice of the chronological marker, how we define time.

Since the sixties the choice induces time frozing. A black hole is supposed to be a collapsing star. In its proper time the time of such collapse is very short : a millisecond. But astrophysicists think that, measured with the time of a distant observer this collpase appears like frozen in time.

It’s very convenient. When people says « what is the final fate of a collapsing neutron star ? » , he replies :

- *As the process extends over an infinite duration I don’t feel obliged to answer that question.*

In a future paper we will suggest a quite different scenario, based on a different time-coordinate choice.

References :

[1] A.Einstein : Erklärung der Perihelbewegung der Merkur aus der allgemeinen Relativitästheorie (Explication de l’avance du périhélie de Mercure par la Relativité Générale). Sitzungsberichte der Königlich Oreubischen Akademie der Wissenchaften (Berlin), 18 nov. 1915. pages 831-839

[2] A.Einstein : Die Fekdgleichungen der Gravitation. Sitzungsberichte der Königlich Oreubischen Akademie der Wissenchaften (Berlin), 25 no. 1915. pages 844-847

[3] K.Schwarzschild : Schwarzschild, K., *Über das Gravitationsfeld eines Massepunktes nach der Einsteinschen Theorie, Preussische Akademie der Wissenschaften, Sitzungberichte*, 1916, p. 189-196.

[4]English translation : On the gravitational field of a mass point according to Einstein's theory http://arxiv.org/pdf/physics/9905030v1.pdf